SECTION 6.2: AREA BETWEEN CURVES

RECALL: If f is continuous on $[a, b]$, and $f(x) \geq 0$ on $[a, b]$, then:

$$\int_a^b f(x)\,dx = \text{the area between the graph of } y = f(x) \text{ and the } x\text{-axis.}$$

More generally, if f is continuous on $[a, b]$, but $f(x) < 0$ for some values of x in $[a, b]$, then:

$$\int_a^b f(x)\,dx = \text{the } \textbf{net area} \text{ between the graph of } y = f(x) \text{ and the } x\text{-axis.}$$

Where 'net area' means the area above the x-axis minus the area below the x-axis.

Suppose f and g are continuous on $[a, b]$ and that $f(x) \geq g(x)$ on $[a, b]$. Then:

$$\int_a^b [f(x) - g(x)]\ dx = \text{the area between the graphs of } y = f(x) \text{ and } y = g(x).$$

NOTE: This formula works just as well for functions of x as it does for functions of y. In general:

$$\text{area between curves} = \int (\text{top curve - bottom curve})\,dx = \int (\text{right curve - left curve})\,dy$$

STRATEGY FOR FINDING THE AREA BETWEEN CURVES:

1. Graph the curves involved.
2. Find the intersection points.
3. Draw a 'representative rectangle' to help you decide between an integral in terms of x or y.

EXAMPLE 1: Find the area, A, of the following regions.

1. The region bounded by $y = 8 - x^2$ and $y = x - 4$.

   Ans: $A = \int_{-4}^{3} \left[(8 - x^2) - (x - 4)\right]\ dx = \ldots = \dfrac{343}{6}$ units^2

2. The region bounded by $y = x$, $y = \dfrac{4}{x}$, $y = 0$, and $x = 4$.

   Ans: $A = \int_0^2 x\,dx + \int_2^4 \dfrac{4}{x}\,dx = \ldots = 2 + 4\ln(2)$ units^2

3. The region between the graphs of $y = \sin(t)$ and $y = \sin(2t)$ for $0 \leq t \leq \pi$.

   Ans: $A = \int_0^{\frac{\pi}{3}} [\sin(2t) - \sin(t)]\ dt + \int_{\frac{\pi}{3}}^{\pi} [\sin(t) - \sin(2t)]\ dt = \ldots = \dfrac{5}{2}$ units^2

4. The area bounded by $y = \pm\sqrt{2x+6}$ and $y = 2x$.

$$\text{Ans: } A = \int_{-2}^{3} \left[\frac{y}{2} - \frac{y^2-6}{2}\right] dy = \ldots = \frac{125}{12} \text{ units}^2$$

5. The area bounded by $y = \sqrt{x}$, $y = 6 - x$, and $y = 0$.

$$\text{Ans: } A = \int_{0}^{2} \left[(6-y) - y^2\right] dy = \ldots = \frac{22}{3} \text{ units}^2$$

CONSUMER AND PRODUCER SURPLUS

In economics, the **demand function**, $p = d(x)$, is the price per item consumers are willing to pay which results in sale of x items. The **supply function**, $p = s(x)$, is the price per item the supplier is willing to accept for the sale of x items. Where the demand and supply curves intersect is called the **equilibrium point**, (x_{eq}, p^*). The **equilibrium price**, p^*, is the price per item at which the supply meets the demand. Algebraically, x_{eq} is the solution to $d(x) = s(x)$ so that $p^* = d(x_{eq}) = s(x_{eq})$. We use Calculus to define two quantities:

- the **consumer surplus at equilibrium**: $C = \displaystyle\int_0^{x_{eq}} (d(x) - p^*)\, dx$

  The number C is the savings to the consumer for purchasing items at the equilibrium price and not higher.

- the **producer surplus at equilibrium**: $P = \displaystyle\int_0^{x_{eq}} (p^* - s(x))\, dx$

  The number P is the benefit to the supplier for selling items at the equilibrium price and not lower.

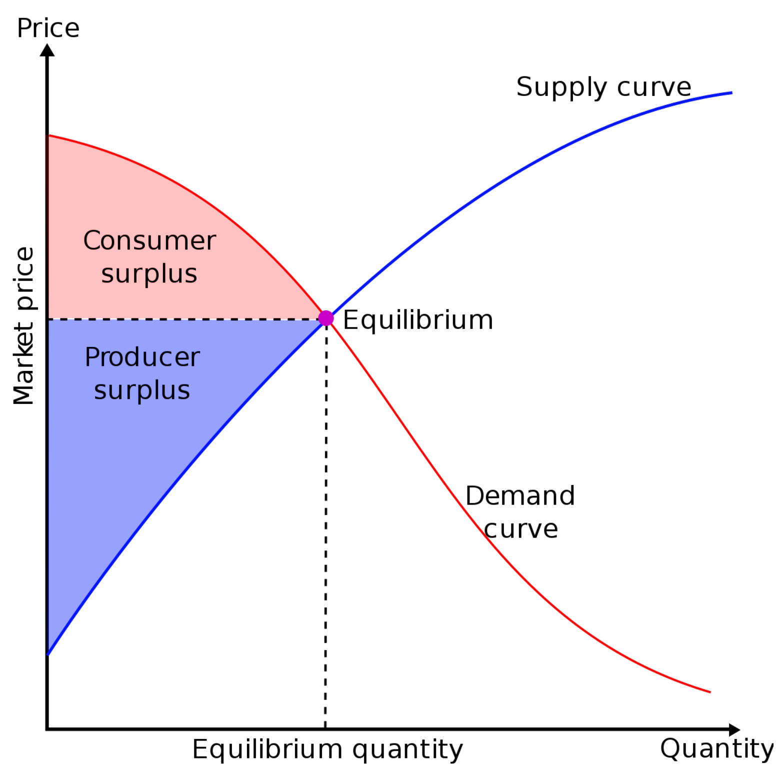

Figure from Wikipedia.org

EXAMPLE 2: The demand and supply functions (in dollars per system) for x game systems are, respectively:

$$d(x) = 400 - 30x, x \geq 0 \quad \text{and} \quad s(x) = x^2, x \geq 0$$

1. Find and interpret $d(5)$.

   Ans: $d(5) = 250$ which means if the price per system is set at \$250, 5 systems will sell.

2. How many systems would be **supplied** if the price of each system were set at \$400?

   Ans: Solving $s(x) = 400$ gives $x = 20$ systems.

3. Find the equilibrium price.

   Ans: Solving $d(x) = s(x)$ gives $x = 10$ systems. $d(10) = s(10) = 100$ so the equilibrium price is \$100.

   (a) How many systems are demanded at this price? Ans: 10 systems.

   (b) How many systems are supplied at this price? Ans: 10 systems.

4. Compute the **consumer surplus** at the equilibrium price.

   Ans: $C = \displaystyle\int_0^{10} [(400 - 30x) - 100]\, dx = ... = 1500$; the consumer surplus is: \$1500.

5. Compute the **producer surplus** at the equilibrium price.

   Ans: $P = \displaystyle\int_0^{10} \left[100 - x^2\right] dx = ... = \frac{2000}{3}$; the producer surplus is approximately \$666.67.

HOMEWORK: Section 6.2: 9 - 64 every other odd; every other odd, 75